Advanced Automatic Control MDP 444

If you have a smart project, you can say "I'm an engineer"

Lecture 8

Staff boarder

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Advanced Automatic Control
MDP 444

- **Lecture aims:**
  - Have a working knowledge of reference inputs, optimal control, and internal model design.

Introduction

Control problem:

$$\Sigma : \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{cases}$$

*Find **stabilizing** control strategy that*

- Minimize objective functional $$J = \int_t^{\infty} F(\mathbf{x}(\tau), \mathbf{u}(\tau))d\tau$$
- Satisfies constraints $$\mathbf{u}(\tau) \in \mathcal{U}, \ \mathbf{x}(\tau) \in \mathcal{X}$$
- is robust towards uncertainty $$\Sigma_0 \in \mathcal{S}$$

Introduction

Control problems:

1 The Minimum Time Control Problem

$$J = \int_{t_0}^{t_f} \mathrm{d}t = t_f - t_0$$

2 The Terminal Control Problem

$$J = [\mathbf{x}(t_f) - \xi(t_f)]^T \mathbf{S} [\mathbf{x}(t_f) - \xi(t_f)]$$

3 The Minimum Control Effort Problem

$$J = \int_{t_0}^{t_f} \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\,\mathrm{d}t$$

4 The Optimal Servomechanism or Tracking Problem

$$J = \int_{t_0}^{t_f} [\mathbf{x}(t) - \xi(t)]^T \mathbf{Q}(t) [\mathbf{x}(t) - \xi(t)]\,\mathrm{d}t = \int_{t_0}^{t_f} \mathbf{e}^T(t)\mathbf{Q}(t)\mathbf{e}(t)\,\mathrm{d}t$$

5 The Optimal Regulator Problem

$$J = \mathbf{x}^T(t_f)\mathbf{S}\mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}(t)^T \mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)]\,\mathrm{d}t$$

Solution strategies

Closed loop optimal control

Feedback: ***u=k(x)***

s.t. closed loop trajectories satisfying optimality

Advantages:

- Feedback
- Uncertainty
- Disturbances
- Unstable systems

Drawbacks

- Find k(x)?

Open loop optimal control

Input trajectory: $u=u(t,x_0)$

solving optimization problem

Advantages:

- Computationally feasible

Drawbacks:

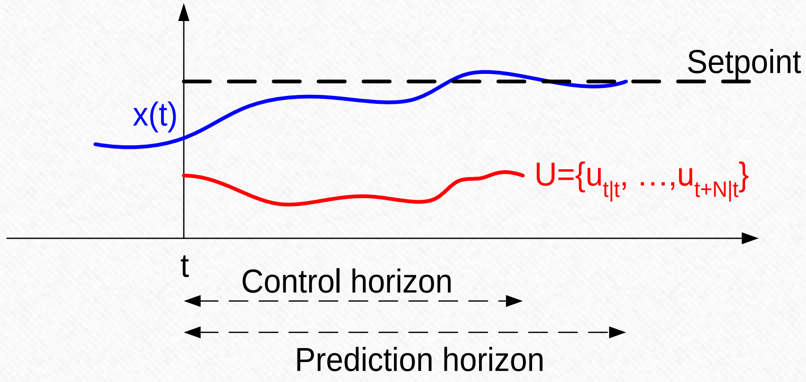
- No feedback
- Disturbances?
- Unstable systems
- Uncertainty

Possible solution 1 : MPC with online optimization

- Solve optimization problem over finite horizon

$$\min_{U \triangleq \{u_{t|t}, u_{t+1|t}, \dots\}} \left\{ J(x(t), U) = \int_{t}^{T_p} F(\mathbf{x}(\tau), \mathbf{u}(\tau)) d\tau \right\}$$

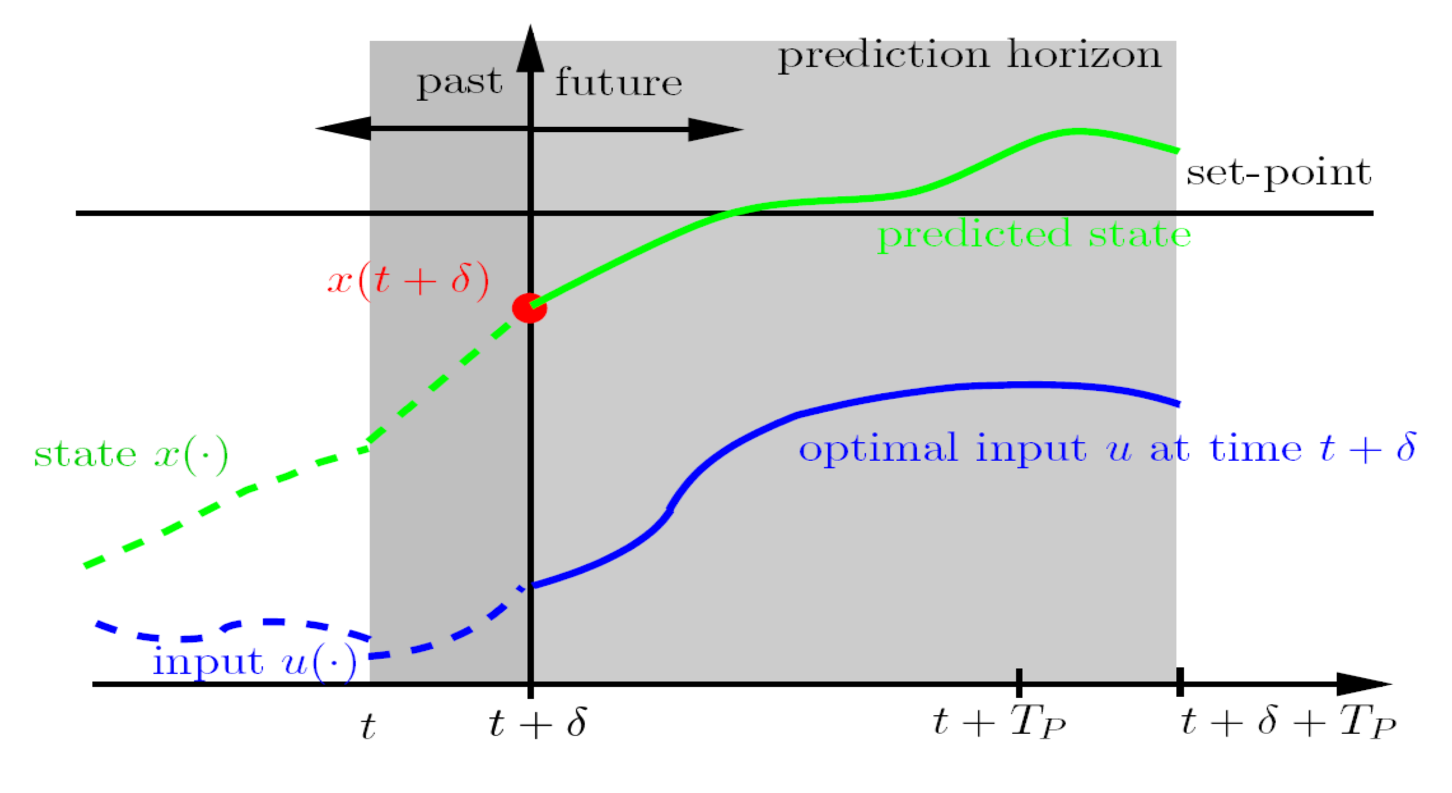
- Implement optimal input for τ2[t,t+δ]



- Re-optimize at next sample (feedback)
- Optimal control inputs implicitly via optimalization

MPC with online optimization

(Allgöwer, 2004)



General Case

- Min/max

$$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{d}t$$

General Case

- Min $$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\, \mathrm{d}t$$
- **Φ** = Endpoint cost
- L =Lagrangian
- u = Control
- X= State

General Case

- Min $$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{d}t$$
- **Φ** = Endpoint cost- final product
- L =Lagrangian
- u = Control
- X= State

General Case

- Min $$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\, \mathrm{d}t$$
- Φ = Endpoint cost- final product
- **L** = Lagrangian – describes dynamics of system
- u = Control
- X= State

General Case

- Min $$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{d}t$$
- Φ = Endpoint cost- final product
- L = Lagrangian – describes dynamics of system
- **u** = Control – what we can do to the system
- X= State

General Case

- Min $$J = \Phi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{d}t$$
- Φ = Endpoint cost- final product
- L = Lagrangian – describes dynamics of system
- u = Control – what we can do to the system
- **X**= State – properties of the system

General Case

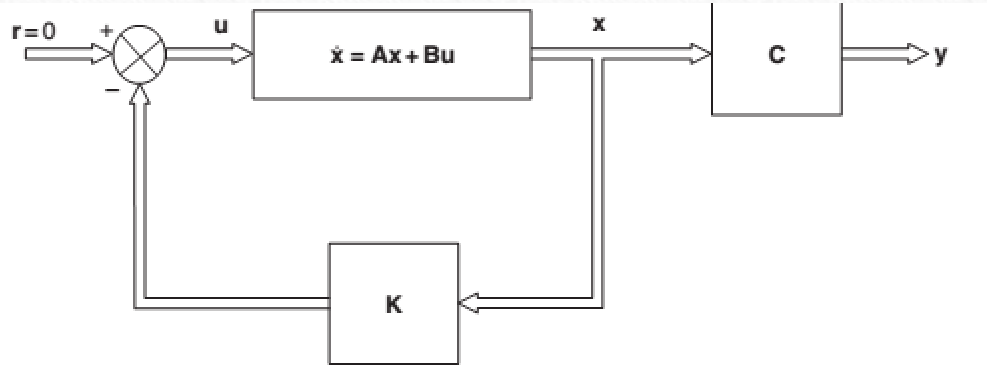
Matrix Riccati Differential Equation

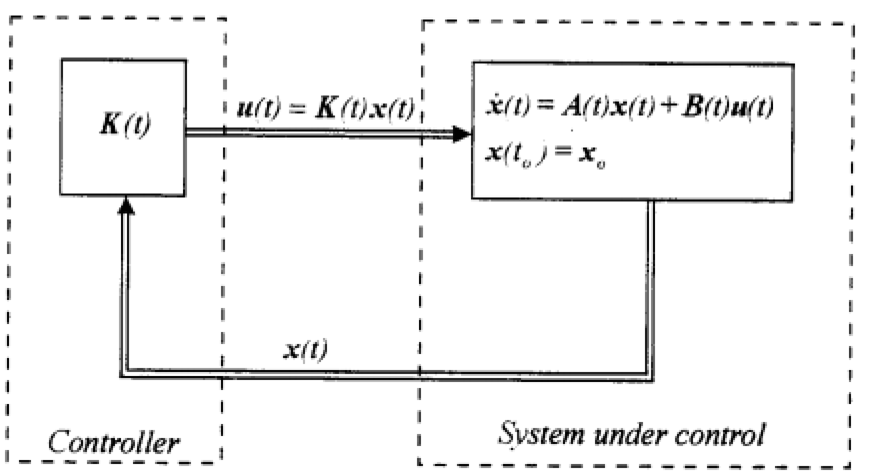
$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) = -\mathbf{Q}(t)$$

The final condition of matrix P(t) and matrix K(t) is called the Kalman matrix

$$\mathbf{u}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)\mathbf{x}(t)$$

$$\mathbf{K}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t)$$



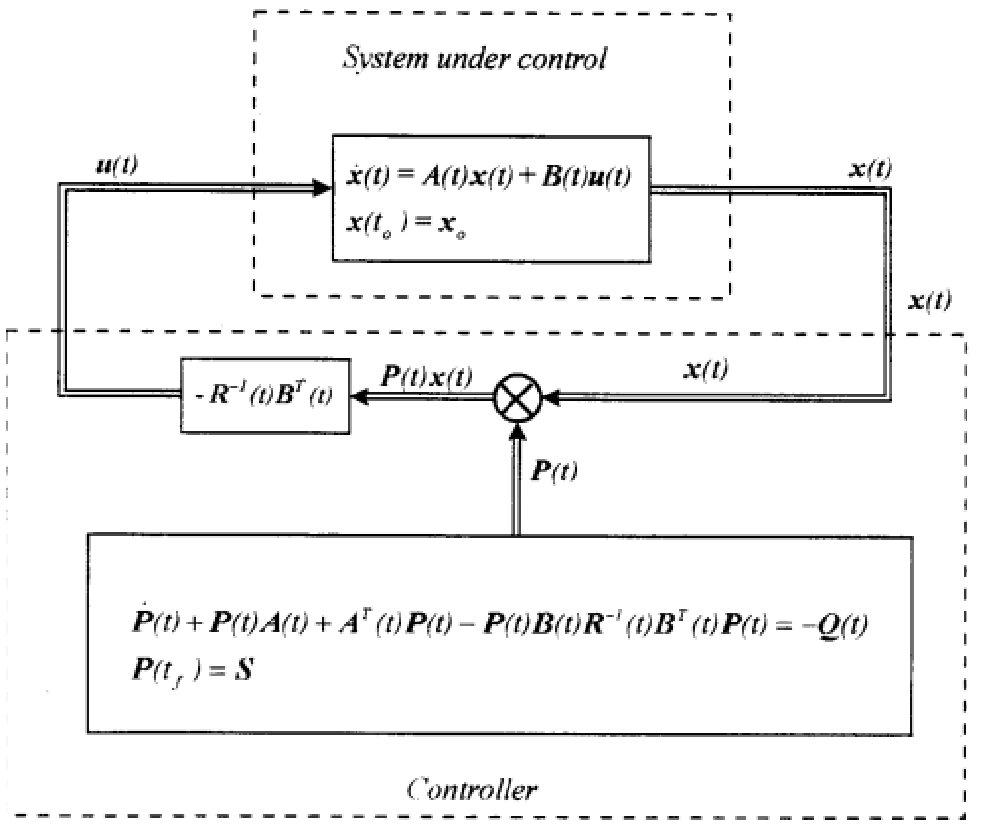


General Case

Matrix Riccati Differential Equation

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{A}^{\mathrm{T}}(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^{\mathrm{T}}(t)\mathbf{P}(t) = -\mathbf{Q}(t)$$

S is positive definite and Q(t) is at least nonnegative definite, or vice versa, and R(t) is positive definite, then a minimum **J** exists if and only if the solution P(t) of the Riccati equation



Discrete-Time State-Space Model

$$x_{k+1} = \mathbf{A}x_k + Bu_k$$

$$y_k = \mathbf{C}x_k$$

The above state-space system is deterministic since no noise is present.

Discrete-Time State-Space Model

We can introduce uncertainty into the model by adding *noise* terms

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

Process noise

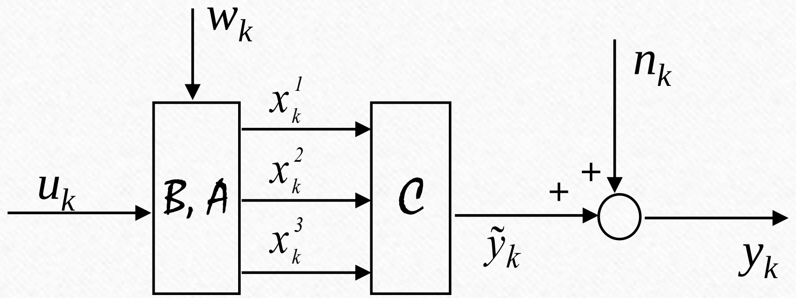
$$y_k = \mathbf{C}x_k + n_k$$

Measurement noise

This is referred to as a stochastic state-space model.

Discrete-Time State-Space Model

This is illustrated below:



X_k - state vector

A - system matrix

B - input matrix

C - output matrix

y_k - output (PV_m)

$\tilde{y}_k$ - noise free output (PV)

w_k - process noise

n_k - measurement noise

u_k - control input (MV)

Observers

We are interested in constructing an optimal observer for the following state-space model:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

An observer is constructed as follows:

$$\hat{x}_{k+1} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k + J(y_k - \hat{y}_k)$$

where J is the observer gain vector, and $\hat{y}_k$ is the best estimate of y_k i.e.

$$\hat{y}_k = \mathbf{C}\hat{x}_k \, .$$

Observers

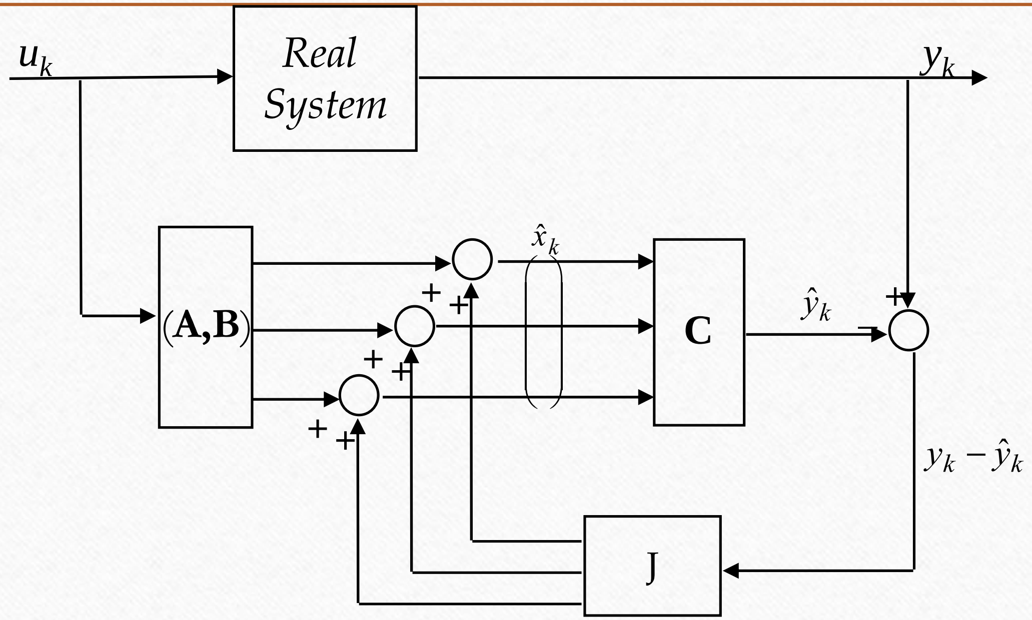
Thus the observer takes the form:

$$\hat{x}_{k+1} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k + J(y_k - \mathbf{C}\hat{x}_k)$$

This equation can also be written as:

$$\hat{x}_{k+1} = (\mathbf{A} - J\mathbf{C})\hat{x}_k + Jy_k + \mathbf{B}u_k$$

Observers



Observer in Block Diagram Form

Kalman Filter

The Kalman filter is a special observer that has *optimal* properties under certain hypotheses. In particular, suppose that.

1) w_k and n_k are statistically independent (*uncorrelated in time and with each other*)
2) w_k and n_k, have Gaussian distributions
3) The system is known exactly

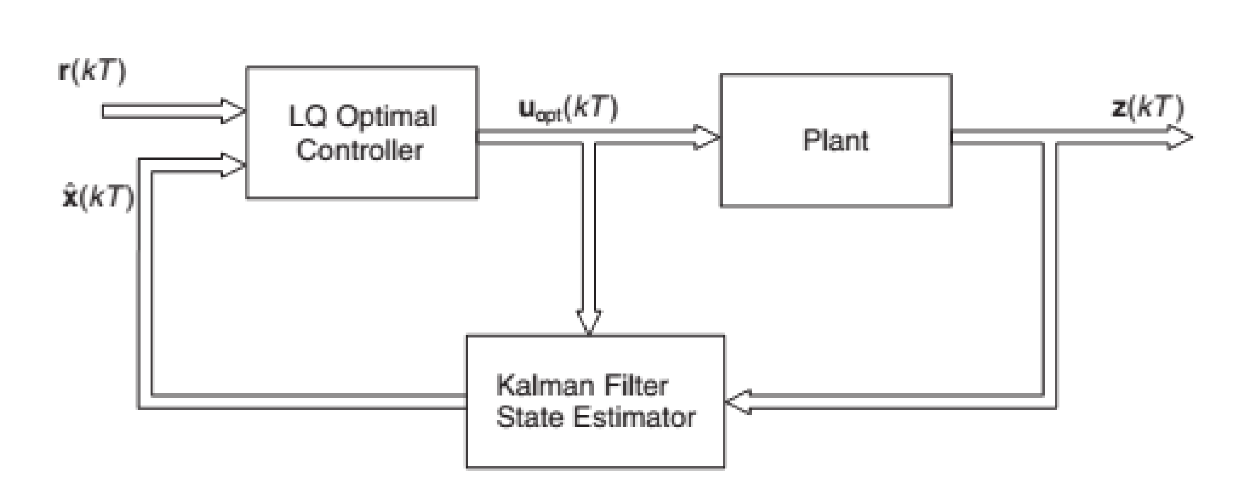
The Kalman filter algorithm provides an observer vector J that results in an optimal state estimate.

Kalman Filter

The optimal *J* is referred to as the Kalman Gain (*J**)

$$\hat{x}_{k+1} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k + J^*(y_k - \hat{y}_k)$$

$$\hat{y} = \mathbf{C}\hat{x}_k$$



Five step Kalman Filter Derivation

Background:

$E[\bullet]$ - Expected Value or Average

$$\Sigma_w^2 = \text{cov}(w_k) = E\left[w_k w_k^T\right] \qquad w_k - \textbf{vector}$$

$$\left(\textbf{scalar}: \sigma_w^2 = \text{var}(w_k) = E\left[w_k^2\right]\right) \qquad \Sigma_w^2 - \textbf{matrix}$$

$$\Sigma_n^2 = \text{cov}(n_k) = E\left[n_k n_k^T\right] \qquad n_k - \textbf{vector}$$

$$\left(\textbf{scalar}: \sigma_n^2 = \text{var}(n_k) = E\left[n_k^2\right]\right) \qquad \Sigma_n^2 - \textbf{matrix}$$

Five step Kalman Filter Derivation

The above assumes w_k and n_k are zero mean. Σ_w^2 and Σ_n^2 are usually diagonal. Σ_w^2 and Σ_n^2 are matrix versions of standard deviation squared or variance.

Five step Kalman Filter Derivation

Step 1:

Given

$$x_{k+1} = \mathbf{A}x_k + w_k$$

$$E[x_0 x_0^T] = P_0$$

$$E[w_k w_k^T] = \Sigma_w^2$$

Calculate

$$P_k = E[x_k x_k^T]$$

Five step Kalman Filter Derivation

Solution:

$$E\left[x_{k+1}x_{k+1}^T\right] = E\left[(\mathbf{A}x_k + w_k)(\mathbf{A}x_k + w_k)^T\right]$$

$$= E\left[(\mathbf{A}x_k + w_k)(x_k^T\mathbf{A}^T + w_k^T)\right]$$

$$= E\left[(\mathbf{A}x_k x_k^T\mathbf{A}^T) + (\mathbf{A}x_k w_k^T) + (w_k x_k^T\mathbf{A}^T) + (w_k w_k^T)\right]$$

$$= \mathbf{A}E\left[x_k x_k^T\right]\mathbf{A}^T + E\left[\mathbf{A}x_k w_k^T\right] + E\left(w_k x_k^T\mathbf{A}^T\right) + E\left[w_k w_k^T\right]$$

$$= \mathbf{A}\mathbf{P}_k\mathbf{A}^T + 0 + 0 + \Sigma_w^2$$

$$\boxed{\mathbf{P}_{k+1} = \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \Sigma_w^2}$$

Five step Kalman Filter Derivation

Step 2:

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$
$$y_k = \mathbf{C}x_k + n_k$$

What is a good estimate of x_k ?

We try the following form for the filter (*where the sequence* $\{J_k\}$ *is yet to be determined*):

$$\hat{x}_{k+1} = \hat{\mathbf{A}}\hat{x}_k + \mathbf{B}u_k + J_k(y_k - \mathbf{C}\hat{x}_k)$$

Five step Kalman Filter Derivation

Step 3:

Given

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

and

$$\hat{x}_{k+1} = \mathbf{A}\hat{x}_k + \mathbf{B}u_k + J_k(y_k - \mathbf{C}\hat{x}_k)$$

Evaluate:

$$\text{cov}(x_k - \hat{x}_k) = E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right]$$

Five step Kalman Filter Derivation

Solution:

$$
\begin{aligned}
\tilde{x}_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\
&= \mathbf{A}x_k + \mathbf{B}u_k + w_k - \left(\mathbf{A}x_k + \mathbf{B}u_k + J_k y_k - J_k \mathbf{C}\hat{x}_k\right) \\
&= \mathbf{A}\tilde{x}_k + w_k - J_k\left(\mathbf{C}x_k + n_k\right) + J_k \mathbf{C}\hat{x}_k \\
&= \mathbf{A}\tilde{x}_k - J_k \mathbf{C}\tilde{x}_k + w_k - J_k n_k \\
&= \left(\mathbf{A} - J_k \mathbf{C}\right)\tilde{x}_k + w_k - J_k n_k
\end{aligned}
$$

Five step Kalman Filter Derivation

Let

$$\mathbf{P}_{k+1} = E\left[\tilde{x}_{k+1}\tilde{x}_{k+1}^T\right]$$

Then applying the result of step 2 we have

$$\mathbf{P}_{k+1} = (\mathbf{A} - J_k\mathbf{C})P_k(\mathbf{A} - J_k\mathbf{C})^T + \sum_w^2 + J_k \sum_n^2 J_k^T$$

Five step Kalman Filter Derivation

Step 4:

Given

$$\mathbf{P_k} = E\left[\tilde{x}_k \tilde{x}_k^T\right]$$

Evolves according to

$$\mathbf{P}_{k+1} = \left(\mathbf{A} - J_k \mathbf{C}\right)\mathbf{P}_k \left(\mathbf{A} - J_k \mathbf{C}\right)^T + \Sigma_w^2 + J_k \Sigma_n^2 J_k^T$$

What is the best (*optimal*) value for J (*call it* $\mathbf{J}_k^*$)?

Five step Kalman Filter Derivation

Solution:

Since $\mathbf{P}_{k+1}$ is quadratic in J_k, it seems we should be able to determine J_k so as to minimize $\mathbf{P}_{k+1}$.

We first consider the scalar case.

$$p_{k+1} = (a - j_k c)^2 \, p_k + \sigma_w^2 + j_k^2 \sigma_n^2$$

The equation for $\mathbf{P}_{k+1}$ then takes the form

Differentiate with respect to j_k

$$\frac{\partial p_{k+1}}{\partial j_k} = -2(a - j_k c) c p_k + 2 j_k \sigma_n^2$$

Hence

$$0 = -(a - j_k c) p_k c + j_k \sigma_n^2$$

Five step Kalman Filter Derivation

Also p_k evolves according to the equation on the top of the slide with j_k replaced by the optimal value j_k*.

$$j_k * = ap_k C\left(Cp_k C + \sigma_n^2\right)^{-1}$$

The corresponding Matrix version is

$$J_k = J_k^* = \mathbf{AP}_k \mathbf{C}^T \left(\mathbf{CP}_k \mathbf{C}^T + \Sigma_n^2\right)^{-1}$$

Five step Kalman Filter Derivation

Step 5:

Bring it all together.

Given

$$x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k + w_k$$

$$y_k = \mathbf{C}x_k + n_k$$

where

$$\Sigma_w^2 = E\left[w_k w_k^T\right]$$

$$\Sigma_w^2 = E\left[n_k n_k^T\right]$$

$$P_0 = E\left[\left(x_0 - \hat{x}_0\right)\left(x_0 - \hat{x}_0\right)^T\right]$$

$\hat{x}_0 =$ Initial state estimate

Find optimal filter.

Five step Kalman Filter Derivation

Solution:

The Kalman Filter

$$\hat{x}_{k+1} = \hat{\mathbf{A}}\hat{x}_k + \mathbf{B}u_k + J_k^*(y_k - \mathbf{C}\hat{x}_k)$$

$$J_k^* = \mathbf{A}\mathbf{P}_k\mathbf{C}^T\left(\mathbf{C}\mathbf{P}\mathbf{C}^T + \Sigma_n^2\right)^{-1}$$

$$\mathbf{P}_{k+1} = \left(\mathbf{A} - J_k^*\mathbf{C}\right)\mathbf{P}_k\left(\mathbf{A} - J_k^*\mathbf{C}\right)^T + \sum_w^2 + J_k^*\sum_n^2 J^{*T}$$

$$= \mathbf{A}\left(\mathbf{P}_k - \mathbf{P}_k\mathbf{C}^T\left(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \sum_n^2\right)^{-1}\mathbf{C}\mathbf{P}_k\right)\mathbf{A}^T + \sum_w^2$$

Five step Kalman Filter Derivation

Example:

The regulator shown in Figure 9.1 contains a plant that is described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$

$$y = [1 \quad 0]\mathbf{x}$$

and has a performance index

$$J = \int_0^\infty \left[\mathbf{x}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{u}^2 \right] dt$$

Determine

(a) the Riccati matrix **P**
(b) the state feedback matrix **K**

Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{R} = \text{scalar} = 1$$

Five step Kalman Filter Derivation

Solution:

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} = \mathbf{0}$$

$$\mathbf{PA} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{21} - 2p_{22} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{P} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - 2p_{21} & p_{12} - 2p_{22} \end{bmatrix}$$

$$\mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 [0 \quad 1] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$= \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} [p_{21} \quad p_{22}]$$

$$= \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}^2 \end{bmatrix}$$

Five step Kalman Filter Derivation

Solution:

$$\begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{21} - 2p_{22} \end{bmatrix} + \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - 2p_{21} & p_{12} - 2p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}^2 \end{bmatrix} = \mathbf{0}$$

Since **P** is symmetric, $p_{21} = p_{12}$

$$-p_{12} - p_{12} + 2 - p_{12}^2 = 0$$

$$p_{11} - 2p_{12} - p_{22} - p_{12}p_{22} = 0$$

$$-p_{22} + p_{11} - 2p_{12} - p_{12}p_{22} = 0$$

$$p_{12} - 2p_{22} + p_{12} - 2p_{22} + 1 - p_{22}^2 = 0$$

Five step Kalman Filter Derivation

Solution:

$$p_{12}^2 + 2p_{12} - 2 = 0$$

solving

$$p_{12} = p_{21} = 0.732 \quad \text{and} \quad -2.732$$

Using positive value

$$p_{12} = p_{21} = 0.732$$

$$2p_{12} - 4p_{22} + 1 - p_{22}^2 = 0$$
$$p_{22}^2 + 4p_{22} - 2.464 = 0$$

solving

$$p_{22} = 0.542 \text{ and } -4.542$$

Using positive value

$$p_{22} = 0.542$$

Five step Kalman Filter Derivation

Solution:

$$p_{11} - (2 \times 0.732) - 0.542 - (0.732 \times 0.542) = 0$$
$$p_{11} = 2.403$$

From equations (9.42), (9.43) and (9.44) the Riccati matrix is

$$\mathbf{P} = \begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix}$$

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} = 1[0 \quad 1]\begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix}$$

$$\mathbf{K} = [0.732 \quad 0.542]$$